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HETERODIFFUSION AND POLYRELAXATION

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Abstract—In a mixture of the main medium with a dispersed filler, the coefficients of mass or energy diffusion of which differ substantially, the exchange by a diffusing substance occurs during the course of the whole process. The averaging of diffusion coefficients for a heterogeneous mixture such as this is impossible. The diffusing substance transport is described by an equation of polyrelaxational type with an infinite spectrum of relaxation times. Three types of polyrelaxational media have been found in which the phase velocity of propagation of high-frequency vibrations increases infinitely, has a constant value or approaches zero with an increase in the angular frequency, i.e. quenching of such oscillations takes place.

NOMENCLATURE

A	amplitude
C	constant
F	Fourier number
H	homochronicity simplex
L	wavelength
M	point
N	dimensionless radius of the point of filler particle
q	mass or energy flux
Q	source, sink
$\bar{r}$	position vector of point M
r_n	relaxation time
$\bar{R}$	dimensionless position vector of point M of the medium
R_n	dimensionless relaxation time
S_0	filler particle surface
T	mass or energy concentration
u^{-1}	radius of convergence of α, β, γ
V_0	filler particle volume
W	phase velocity
x	abscissa
∇^2	Laplacian.

Greek symbols

α, β	power series
γ	combination of power series
θ	dimensionless concentration
Λ	dimensionless wavelength
ξ	dimensionless abscissa
Π	volumetric fraction of filler
$\bar{\rho}$	position vector of the inner point of a particle
ρ_0	filler particle radius
τ	time
ω	angular frequency.

1. POLYRELAXATION

The terminal velocity of diffusing substance propagation may owe its origin to two different reasons: either to the dependence of the diffusion coefficient D on substance concentration T, or to the relaxation properties of the medium. It is not improbable that the reason for the terminal nature of phase velocity is the combination of medium nonlinearity with its relaxation. In the first case, the diffusion equation turns to be a nonlinear parabolic one and its integration is often possible only by numerical means. In the second case, the transport equation turns to be a linear hyperbolic one, for which an analytical solution has been found in many important cases [1].

It is pertinent to note that under certain conditions the unique solutions of the hyperbolic heat conduction equation contradict the second law of thermodynamics [2]. It may well be that there are relaxation time spectra at which the solutions of the transport equation correspond to the second law for all the boundary-value and initial conditions, but this has not been generally proved.

The relaxation time and thermal diffusivity of the substance were determined [3] by conducting prolonged measurements of temperature [4]. The time of diffusional relaxation was found from the measurements of the barycentre moment [5].

When a substance has the relaxation time spectrum

$$r_1 > |r_2| > |r_3| > \cdots \geqslant u \geqslant 0, \tag{1}$$

then the concentration gradient T can be disposed with respect to the time, τ, derivatives of the flow, q, as

$$-D\nabla T = \bar{q} + r_1 \frac{\partial \bar{q}}{\partial \tau} + r_2^2 \frac{\partial^2 \bar{q}}{\partial \tau^2} + \cdots \tag{2}$$

In some cases, the physico-geometric parameters of an inhomogeneous medium, r_n^n, which have the dimensions of the powers of time, can be negative (from the second parameter onwards), so that the very quantities r_n are positive, negative or pure imaginary, depending on the sign of r_n^n and parity of the power n. The conditions, which are imposed on the averaging matrix [6], do not imply whether all of its coefficients are positive or real.

It should be emphasized that according to the second law of thermodynamics, to the zero value of the energy or mass concentration gradient, ∇T

$$\nabla T(M_0, \tau_0) = 0, \tag{3}$$

there corresponds, at point M_0 of a homogeneous medium at time τ_0, the zero value of the flux of transported substance

$$\bar{q}(M_0, \tau_0) = 0. \tag{4}$$

Therefore, in the vicinity of the spatial-temporal point (M_0, τ_0), the flux will satisfy the homogeneous equation

$$r_1 \frac{\partial \bar{q}}{\partial \tau} + r_2^2 \frac{\partial^2 \bar{q}}{\partial \tau^2} + \cdots = 0, \tag{5}$$

and also the homogeneous boundary-value condition, equation (4). The solutions of this equation, for which equation (4) is not fulfilled, are not consistent with the second law of thermodynamics.

The energy or mass conservation equation for such a polyrelaxational medium

$$\frac{\partial T}{\partial \tau} + r_1 \frac{\partial^2 T}{\partial \tau^2} + r_2^2 \frac{\partial^3 T}{\partial \tau^3} + \cdots = D\nabla^2 T, \tag{6}$$

is linear, provided the mass or energy diffusion coefficient D is independent of the substance transferred [2].

The transport properties of the polyrelaxational medium should not depend on the shape and dimensions of the body, and most conveniently these can be considered by taking as an example the object of simplest geometry; therefore further conclusions relate to the half-space $x \in [0, \infty]$, with periodic changes of concentration occurring on its surface

$$T(0, \tau) = A \cos \omega\tau. \tag{7}$$

In this case, the concentration far from the surface does not vary

$$T(\infty, \tau) = 0. \tag{8}$$

It is worthwhile to determine the relative concentration θ, abscissa ξ, time H and, which is most important, the spectrum of dimensionless relaxation times R_n with the aid of the relations

$$\theta = T/A; \quad \xi = \frac{x}{\sqrt{(\omega/D)}}; \quad H = \omega \cdot \tau; \quad R_n = \omega \cdot r_n, \tag{9}$$

where, similarly to equation (1)

$$R_1 > |R_2| > |R_3| > \cdots > |R_n| > \cdots \geqslant U \geqslant 0. \tag{10}$$

After such substitutions, the transport equation

$$\sum_{n=0}^{\infty} R_n^n \frac{\partial^{n+1}\theta}{\partial H^{n+1}} = \frac{\partial^2 \theta}{\partial \xi^2}, \tag{11}$$

and the boundary conditions of the first kind

$$\theta(0, H) = \cos H; \quad \theta(\infty, H) = 0, \tag{12}$$

can be written in a simpler form.

The polyrelaxational conservation equation (11) has the solution at all the values of time, including such large ones when the initial distribution of concentration can be neglected. This asymptotic solution has the form

$$\theta(N, H) = \exp\left[-\frac{\xi}{\sqrt{2}}\sqrt{(\sqrt{(\alpha^2+\beta^2)}-\alpha)}\right] \times \cos\left[H - \frac{\xi}{\sqrt{2}}\sqrt{(\sqrt{(\alpha^2+\beta^2)}+\alpha)}\right], \tag{13}$$

where the parameters α and β are the power series

$$\alpha(\omega) = \sum_{m=1}^{\infty} (-1)^{m-1} R_{2m-1}^{2m-1} = \sum_{m=1}^{\infty} (-1)^{m-1} \omega^{2m-1} r_{2m-1}^{2m-1}, \tag{14}$$

$$\beta(\omega) = \sum_{m=0}^{\infty} (-1)^m R_{2m}^{2m} = \sum_{m=0}^{\infty} (-1)^{2m} r_{2m}^{2m}, \tag{15}$$

with the common radius of convergence

$$\lim_{n\to\infty} \frac{1}{(r_n^n)^{1/n}} = \frac{1}{u} \leqslant \infty. \tag{16}$$

It is clear that the neighbouring points ξ and $\xi + \Lambda$ are inphase when the dimensionless wavelength Λ is

$$\Lambda = 2\pi\sqrt{2}/\sqrt{(\gamma(\omega))}, \tag{17}$$

where

$$\gamma(\omega) = \alpha(\omega) + \sqrt{(\alpha^2(\omega) + \beta^2(\omega))}. \tag{18}$$

The actual physical value of the wavelength L is

$$L = \Lambda\sqrt{(D/\omega)} = 2\pi\sqrt{(2D/\gamma(\omega))}. \tag{19}$$

The phase velocity of propagation of vibrations

$$W(\omega) = L \cdot \omega/\sqrt{(2\pi)} = \sqrt{(2\omega D/\gamma(\omega))}. \tag{20}$$

depends substantially on the nature of the relaxation time spectrum.

But, since for any nature of the spectrum function (18) at the zero frequency is

$$\gamma(0) = 1, \tag{21}$$

then in any polyrelaxational medium, including also a completely nonrelaxational one, to the zero frequency there corresponds the zero value of phase velocity

$$W(0) = 0. \tag{22}$$

Thus, if the medium is completely nonrelaxational, then $\alpha(\omega) = 0$, $\beta(\omega) = 1$, $\gamma(\omega) = 1$, and the phase velocity

$$W_0(\omega) = \sqrt{(2\omega D)} \tag{23}$$

increases infinitely with frequency. This circumstance is responsible for the hyperbolic heat conduction equations

$$\frac{\partial T}{\partial \tau} + r_1 \frac{\partial^2 T}{\partial \tau^2} = D\nabla^2 T, \tag{24}$$

having been considered by Cattaneo [7] and Tavernier [8]. For their monorelaxational medium function (18) takes the form

$$\gamma(\omega) = \omega r_1 + \sqrt{(1 + \omega^2 r_1^2)}, \tag{25}$$

and the phase velocity

$$W_1(\omega) = \sqrt{(2\omega D/[\omega r_1 + \sqrt{(1+\omega^2 r_1^2)}])}, \tag{26}$$

has the well-known [7–10] terminal value

$$\lim_{\omega\to\infty} W_1(\omega) = \sqrt{(D/r_1)}. \tag{27}$$

If the medium is birelaxational, then the LHS of equation (6) contains three terms

$$\frac{\partial T}{\partial r} + r_1\frac{\partial^2 T}{\partial \tau^2} + r_2^2\frac{\partial^3 T}{\partial \tau^3} = D\nabla^2 T. \tag{28}$$

Function (18) represents the sum

$$\gamma(\omega) = r_1\omega + \sqrt{((1-\omega^2 r_2^2) + r_1^2\omega^2)}. \tag{29}$$

The phase velocity of the propagation of vibrations, equation (20)

$$W_2(\omega) = \sqrt{(2\omega D/[r_1\omega + \sqrt{((1-\omega^2 r_2^2)^2 + r_1^2\omega^2)}])}. \tag{30}$$

at zero and infinitely large frequencies vanishes

$$W_2(0) = W_2(\infty) = 0 \tag{31}$$

and at the frequency $\omega_m = 1/r_2$ takes the maximum value

$$\max W_2(\omega_m) = \sqrt{(D/r_1)}. \tag{32}$$

If the relaxation time spectrum contains a finite number of terms v, and $v > 2$, then series (14) and (15) consist of polynomials in the frequency ω. At high frequencies, the function $\gamma(\omega)$ increases proportionally to ω^v, and with the increase in the frequency, the phase velocity in such a polyrelaxational medium becomes negligibly small

$$\lim_{\omega\to\infty} W_v(\omega) = 0. \tag{33}$$

It may well be that in view of the zero values of phase velocity, equation (20), at small, equation (22), at very large, equation (33), frequencies, ω, the velocity $W_v(\omega)$ has more than one extremum which depends on the relaxation time ratios.

Consequently, in a polyrelaxational medium, high velocity of propagation is typical of low-frequency vibrations, while high-frequency vibrations are quenched in such a medium. For example, if the concentration on the surface can be represented by a Fourier series

$$T(0,\tau) = \sum_{n=1}^{\infty} A_n \cos\omega_n\tau, \quad \omega_n = n\cdot\omega_1, \tag{34}$$

then an oscillating field appears inside of the half-space

$$T(x,\tau) = \sum_{n=1}^{\infty} A_n \exp\left[-x\sqrt{\left(\frac{\omega_n}{2D}\right)}\sqrt{(\gamma(\omega_n) - 2\alpha(\omega_n))}\right]$$

$$\times\cos\left[\omega_n\tau - x\sqrt{\left(\frac{\omega_n}{2D}\Big/\gamma(\omega_n)\right)}\right], \tag{35}$$

the harmonic phases of which propagate with the velocities

$$W(\omega_n) = \sqrt{(2\omega_n D/\gamma(\omega_n))}, \tag{36}$$

which decrease with an increasing frequency

$$\lim_{n\to\infty} W(\omega_n) = 0. \tag{37}$$

Starting from v, the following inequalities hold for the phase velocities

$$W(\omega_v) > W(\omega_{v+1}) > W(\omega_{v+2}) > \cdots \tag{38}$$

It is the highest phase velocity $W(\omega_v)$ which is the velocity of propagation of disturbances in the half-space. It may well be, although it has not been proved, that these relations hold starting from the first harmonic $v = 1$, which propagates with the highest velocity $W(\omega_1)$.

When the relaxation time spectrum, equation (1), consists of an infinite number of terms, then the value of $W(\omega)$ depends on the behaviour of the function $\gamma(\omega)$. In this case, when the radius of convergence, equation (16), of series (14) and (15) is infinite, the velocity $W(\omega)$ depends on the behaviour of $\gamma(\omega)$. If asymptotically the following relation holds for the function $\gamma(\omega)$

$$\gamma(\omega) = C\cdot\omega^s + D\left(\frac{1}{\omega}\right), \tag{39}$$

then the limiting value of the phase velocity is determined as

$$\lim_{\omega\to\infty} W(\omega) = \begin{cases} \infty, & s < 1; \\ \text{const.}, & s = 1; \\ 0, & s > 1. \end{cases} \tag{40}$$

When the convergence radius, equation (16), is finite, then at all the frequencies

$$\omega_m > u^{-1}, \tag{41}$$

the phase velocity $W(\omega_m)$ is strictly equal to zero

$$W(\omega_m) = 0. \tag{42}$$

An appreciable propagation velocity is typical of the harmonics with the frequency

$$\omega_\mu < \omega_m; \tag{43}$$

and in this case a 'smooth' signal on the surface, equation (34), is transformed into a 'rough' oscillating signal inside of the half-space, equation (35), containing a finite number of components.

2. POLYRELAXATION AND INITIAL CONDITIONS

The concentration field in a relaxational medium is a function not only of the abscissa x and time τ, but also depends parametrically on the relaxation time spectrum, i.e.

$$T = T(x,\tau; r_1, r_2, \ldots). \tag{44}$$

For a nonrelaxational medium

$$T = T(x,\tau). \tag{45}$$

A retarding effect of relaxation is evidenced physically by the fact that concentration variations inside of a body tend to be less noticeable than on its surface. At the initial time, when the effect of the body surface state is as yet not felt at all at the inner points, the following inequalities take place

$$\left|\frac{\partial^p T(x,0;\,r_1,r_2,\ldots)}{\partial\tau^p}\right| \leqslant \left|\frac{\partial^p T(x,0)}{\partial\tau^p}\right|, \quad p = 1,2,3,\ldots \tag{46}$$

But, as it follows from the diffusion equation, in a nonrelaxational medium at the initial time

$$\frac{\partial T(x,0)}{\partial\tau} = DT_0''(x), \tag{47}$$

where $T_0(x)$ is the initial distribution of concentrations. In general

$$\frac{\partial^p T(x,0)}{\partial\tau^p} = D^p T_0^{(2p)}(x). \tag{48}$$

Therefore, for a polyrelaxational medium, inequality (46) yields

$$\left|\frac{\partial^p T(x,0;\,r_1,r_2,\ldots)}{\partial\tau^p}\right| \leqslant D^p|T_0^{(2p)}(x)|. \tag{49}$$

If the initial distribution is uniform

$$T_0(x) = T_0 = \text{const.}, \tag{50}$$

then all $T_0^{(2p)}(x) \equiv 0$, and for a polyrelaxational medium the initial conditions are simply formulated as

$$\frac{\partial^p T(x,0;\,r_1,r_2,\ldots)}{\partial\tau^2} = 0; \quad p = 1,2,3,\ldots \tag{51}$$

If the initial distribution is nonuniform, then, as it follows from equation (46), the following initial conditions are calculated

$$\sum_{n=0}^{\infty} r_n^n \frac{\partial^{n+1+p}}{\partial\tau^{n+1+p}} T(x,0;\,r_1,r_2,\ldots) = D^p T_0^{(2p)}(x), \quad p = 0,1,2,3,\ldots \tag{52}$$

3. HETERODIFFUSION

Diffusion in a nonhomogeneous medium, the components of which exchange mass during the whole course of the process, cannot be described by the equation which involves an averaged diffusion coefficient [10]. For example, sugar migration between the cells of a living being and the blood is not amenable to treatment as a process of sugar diffusion in some averaged material. A mathematical model of diabetes requires that the diffusion of sugar in cells and in the blood and also sugar exchange between these components of the organism are taken into account. Completely similar phenomena are observed in nonsaturated solid solutions [11], in unsteady-state heating of granular massif [12]. Therefore, to impart generality, a topologically multibound medium can be called the main medium with the diffusion coefficient D_e; singly bound dispersed inclusions can be called the filler with the diffusion coefficient D_i. The filler acts on the main medium as an accumulator, which absorbs mass or energy when its concentration in the main medium increases, and releases it when concentration in the medium decreases. The filler prevents a change of concentration in the external medium $T_e(M,\tau)$ at each point M and at each time τ. Therefore, when the main medium or phase becomes saturated, i.e.

$$\partial T_e(M,\tau)/\partial\tau > 0, \tag{53}$$

then the filler absorbs Q units of diffusant per unit volume and per unit time. The distribution of concentrations in the main medium satisfies the conservation equation

$$\frac{\partial T_e}{\partial\tau} = D_e \nabla_e^2 T_e - Q. \tag{54}$$

As a rule, the geometry of the main medium is poorly associated with the configuration of filler particles, and in orthogonal and curvilinear coordinates (x^1, x^2, x^3) the Laplacian for this medium has a standard form [13]

$$\nabla_0^2 T = \frac{1}{\sqrt{g}} \sum_{j=1}^{3} \frac{\partial}{\partial x}\left(\frac{\sqrt{g}}{g_j}\frac{\partial T_e}{\partial x}\right), \tag{55}$$

where g_j are the Lame parameters and $\sqrt{g}$ is the Jacobian of the system.

A plus (+) sign should be inserted in front of Q if concentration in the main phase would have decreased

$$\partial T_e(M,\tau)/\partial\tau < 0, \tag{56}$$

and the amount of diffusant in the filler would have made up in some way for its decrease in the external medium. For the mean volume of filler particles $V_0(\mathrm{m}^3)$, their quantity per unit of volume $n(\mathrm{m}^{-3})$, the volumetric fraction of the filler is equal to

$$\Pi = n \cdot V_0. \tag{57}$$

The radius of the filler particles is expressed in terms of their surface S_0 and volume V_0 as

$$\rho_0 = 3V_0/S_0, \tag{58}$$

when the inclusions are three-dimensional. When the inclusions of a dispersed phase are formed by filaments with the average cross-sectional area F_0 and perimeter P_0, then the radius of such elongated two-dimensional particles is defined as

$$\rho_0 = 2F_0/P_0. \tag{59}$$

The position vector of the points on the surface of filler particles, $\bar{\rho}_0$, can be written as the sum of the vector $\bar{r}$, corresponding to the particle centre, and of the product of the vector $\bar{\rho}_0$ by the tensor E

$$\bar{r}_0 = \bar{r} + \bar{\rho}_0 \cdot E, \tag{60}$$

satisfying the conditions

$$|\bar{\rho}_0 \cdot E| = |\bar{\rho}_0| = \rho_0. \tag{61}$$

The radii of the filler particles, ρ_0, are so small, that the diffusant concentration on their surface in the external medium remains practically unchanged

$$T_e(\bar{r}+\bar{\rho}_0 \cdot E, \tau) \simeq T_e(\bar{r}, \tau). \tag{62}$$

On the other hand, the energy or mass concentration at the inner point of the filler is determined not only by the vector

$$\bar{\rho} = \bar{\rho}_0 \cdot E, \quad 0 \leqslant \eta \leqslant 1, \tag{63}$$

which emanates from the particle centre, but also depends parametrically on the vector $\bar{r}$, which characterizes the position of this centre. Therefore, the concentration inside of a particle $T_i(\bar{\rho}, \tau, \bar{r})$ has $\bar{\rho}$, τ and $\bar{r}$ as arguments. It satisfies the mass or energy diffusion conservation equation in a spherical cylindrical coordinate system

$$\frac{\partial T_i}{\partial \tau} = D_i \left[\frac{\partial^2 T_i}{\partial \rho^2} + \frac{k_i - 1}{\rho} \frac{\partial T_i}{\partial \rho} \right], \tag{64}$$

the centre of which is determined by the vector $\bar{r}$. Here, D_i($m^2 s^{-1}$) is the mass or energy diffusion coefficient, $k = 3, 2, 1$ is the number of final measurements of filler particles. The concentration of diffusant at the particle centre is always extremal

$$\frac{\partial T_i(0, \tau, \bar{r})}{\partial \rho} = 0. \tag{65}$$

At the start of the process, it may depend on the centre position

$$T_i(\bar{\rho}, 0, \bar{r}) = T_{i0}(\bar{r}) = \text{const.} \tag{66}$$

and is assumed to be uniformly distributed over the particle volume. It differs from the initial concentration $T_e(\bar{r}, 0)$ of the diffusant in the main phase

$$T_{i0}(\bar{r}) \neq T_e(\bar{r}, 0) = T_{e0} = \text{const.} \tag{67}$$

At the end of the process, when the diffusant concentrations in the main and dispersed phases attain their equilibrium values $T_{e\infty}$ and $T_{i\infty}$, these latter can also differ

$$T_e(\bar{r}, \infty) \neq T_i(\bar{\rho}, \infty, \bar{r}). \tag{68}$$

In the case of heat transfer, this difference is attributed to different volumetric heat capacities c_e and c_i($J\ m^{-3}\ K^{-1}$) of the main and dispersed phases, although their temperatures will equalize to the end.

However, if one determines the dimensionless concentrations of the main phase

$$\theta_e(\bar{r}, \tau) = [T_e(\bar{r}, \tau) - T_{e0}]/[T_{e\infty} - T_{e0}], \tag{69}$$

and of the filler

$$\theta_i(\bar{\rho}, \tau, \bar{r}) = [T_i(\bar{\rho}, \tau, \bar{r}) - T_{i0}]/[T_{i\infty} - T_{i0}], \tag{70}$$

then at the start and at the end of the process these values will be the same

$$\theta_e(\bar{r}, 0) = \theta_i(\bar{\rho}, 0, \bar{r}) = 0, \tag{71}$$

$$\theta_e(\bar{r}, \infty) = \theta_i(\bar{\rho}, \infty, \bar{r}) = 1. \tag{72}$$

This allows an assumption that in a linear approximation the equality of dimensionless concentrations on the surface of filler particles holds during the entire course of exchange

$$\theta_i(\rho_0, \tau, \bar{r}) = \theta_e(\bar{r}, \tau). \tag{73}$$

When the initial temperatures of the main phase and of the filler coincide, this equality is very accurately fulfilled.

Thus, the relative concentration of diffusant in the filler particles satisfies the boundary conditions

$$\begin{gathered} \frac{\partial}{\partial \rho} \theta_i(0, \tau, \bar{r}) = 0, \\ \theta_i(\bar{\rho}, 0, \bar{r}) = 0; \quad \theta_i(\rho_0, \tau, \bar{r}) = \theta_e(\bar{r}, \tau) \end{gathered} \tag{74}$$

and, of course, the conservation law

$$(D_i)^{-1} \frac{\partial \theta_i}{\partial \tau} = \frac{\partial^2 \theta_i}{\partial \rho^2} + \frac{k-1}{\rho} \frac{\partial \theta_i}{\partial \rho}. \tag{75}$$

The solution of the problem, equations (74) and (75), is known [2, 9, 13, 14] and is expressed in terms of the Borel convolution

$$\theta_i(\rho, \tau, \bar{r}) = \int_0^T \theta_e(\bar{r}, \tau - t) G(\rho, t)\, \mathrm{d}t \tag{76}$$

of the dimensionless concentration of the main medium $T_e(\bar{r}, \theta)$ and of Green's function

$$G(\rho, t) = -\frac{D_i}{\rho_0^2} \sum_{n=1}^{\infty} \frac{2\mu_n \cos(\mu_n \rho\, \rho_0^{-1})}{\cos' \mu_n} \exp\left[-\mu_n \frac{D_i t}{\rho_0^2} \right]. \tag{77}$$

The generalized cosine is the series [2, 15, 16]

$$\cos \mu = 1 - \frac{\mu^2}{k \cdot 2} + \frac{\mu^4}{k(k+2) 2 \cdot 4} - \cdots, \tag{78}$$

which has an infinite radius of convergence at all positive k and a radius, which coincides with the trigonometric cosine at $k = 1$. The characteristic numbers μ_p are the roots of this transcendental function

$$\cos \mu = 0. \tag{79}$$

Physically, the substance sink $Q(\bar{r}, \tau)$ is the quantity of diffusant, which penetrates from the main medium into the particles of the dispersed phase through the total surface of filler particles per unit volume. The diffusant flux through a particle surface is

$$q = -D_i [\partial T_i(\rho, \tau, \bar{r})/\partial \rho]_{\rho = \rho_0}. \tag{80}$$

The value of the diffusant sink is proportional to the surface area of particles per unit volume $n \cdot S_0$ and the diffusant flux. Depending on the number of final measurements, it is equal to:

$$Q = \frac{\Pi \cdot S_0}{V_0} q = \frac{3\Pi}{\rho_0} q, \quad k = 3; \tag{81}$$

$$Q = \frac{\Pi \cdot P_0}{F_0} q = \frac{2\Pi}{\rho_0} q, \quad k = 2; \tag{82}$$

since the quantity of particles per unit volume is Π/V_0 or

Π/F_0. In view of expression (80) for the flux, the sink is

$$Q = -\frac{k\Pi}{\rho_0} D_i \frac{\partial}{\partial \rho} T_i(\rho_0, \tau, \bar{r}). \tag{83}$$

Therefore, the conservation equation (54) can be represented in the form

$$\frac{\partial T_e}{\partial \tau} = D_e \nabla_e^2 T_e + \frac{k\Pi}{\rho_0} D_i \frac{\partial}{\partial \rho} T_i(\rho_0, \tau, \bar{r}), \tag{84}$$

as if it includes the field of concentrations inside of the particles of dispersed phase $T_i(\rho, \tau, \bar{r})$. But the distribution of concentrations in the particles of a filler depends completely on the field of concentrations in the surrounding medium, equation (76)

$$\frac{T_i(\rho, \tau, \bar{r})}{T_{i\infty} - T_{i0}} = \frac{1}{T_{e\infty} - T_{e0}} \int_0^\tau [T_e(\bar{r}, \tau - t) - T_{e0}] G(\rho, t)\, dt, \tag{85}$$

so that virtually the conservation law, equation (54), for the main medium

$$\frac{\partial T_e}{\partial \tau} = D_e \nabla_e^2 T_e - \frac{k\Pi}{\rho_0} D_i \frac{T_{i\infty} - T_{i0}}{T_{e\infty} - T_{e0}} \times \frac{\partial}{\partial \rho} \int_0^\tau [T_e(\bar{r}, \tau - t) - T_{e0}] [G(\rho, t)]_{\rho = \rho_0} \, dt \tag{86}$$

is an integro-differential equation for the field of concentrations $T_e(\bar{r}, \tau)$. With equation (77) for the Green's function taken into account, equation (86) will take on a more detailed form

$$\frac{\partial T_e}{\partial \tau} = D_e \nabla^2 T_e - \frac{k\Pi}{\rho_0} D_i \frac{T_{i\infty} - T_{i0}}{T_{e\infty} - T_{e0}} \frac{\partial}{\partial \rho} \times \int_0^\tau [T_e(\bar{r}, \tau - t) - T_{e0}] \times \sum_{p=1}^{\infty} \left[\frac{2\mu_p \cos(\mu_p \rho \rho_0^{-1})}{\cos' \mu_p} \right]_{\rho = \rho_0} \times \exp\left(-D_i \frac{t}{\rho_0^2} \right) \frac{D_i \, dt}{\rho_0^2}. \tag{87}$$

For further analysis, it is advisable to determine the dimensionless arguments and parameters

$$F = \frac{D_i \tau}{\rho_0^2}; \quad \phi = \frac{D_i \theta}{\rho_0^2}; \quad N = \frac{\rho}{\rho_0}; \quad \bar{R} = \frac{\bar{r}}{\rho_0}; \quad G = \frac{T_{i\infty} - T_{i0}}{T_{e\infty} - T_{e0}}; \quad \Delta = \frac{D_e}{D_i}; \tag{88}$$

and to multiply all the terms of equation (87) by the dimensional parameter $\rho_0^2 D_i^{-1} [T_{e\infty} - T_{e0}]^{-1}$ to yield

$$\frac{\partial \theta_e}{\partial F} = \nabla_{\bar{R}}^2 \theta_e - k\Pi\sigma \frac{\partial}{\partial N} \int_0^F \theta_e(\bar{R}, F - \phi) \times \sum_{p=1}^{\infty} \left[\frac{2\mu_p \cos \mu_p N}{\cos' \mu_p} \right]_{N=1} \cdot \exp(-\mu_p^2 \phi) \, d\phi. \tag{89}$$

The Borel convolution of the analytical function $T_e(\bar{R}, F)$ and of the series

$$G(N, F) = -\sum_{p=1}^{\infty} \frac{2\mu_p \cos \mu_p N}{\cos' \mu_p} \exp(-\mu_p^2 F), \tag{90}$$

was considered in detail in ref. [2]. It describes the distribution of dimensionless concentrations inside of the filler particles

$$\theta_i(N, F, \bar{R}) = \int_0^F \theta_e(\bar{R}, F - \phi) G(N, \phi) \, d\phi, \tag{91}$$

parametrically dependent on the position of the particle centre with vector $\bar{r}$. According to ref. [2], the Duhamel integral, equation (91), can be written as a sum of the components of effect and after effect

$$\theta_i(N, F, \bar{R}) = \theta_a(N, F, \bar{R}) + \theta_r(N, F, \bar{R}). \tag{92}$$

The component of the effect is the series

$$\theta_a(N, F, \bar{R}) = \sum_{n=0}^{\infty} \frac{\partial^n}{\partial F^n} \theta_e(\bar{R}, F) P_n(N, k), \tag{93}$$

in terms of the actual values of time derivatives of concentration and radial polynomials of the symmetric field $P_n(N, k)$. The after effect component is also the series

$$\theta_r(N, F, \bar{R}) = \sum_{n=1}^{\infty} \frac{\partial^n}{\partial F^n} \theta_e(\bar{R}, 0) \cdot P_n(N, F, k), \tag{94}$$

in terms of the initial values of these partial time derivatives $\partial^n \theta_e(\bar{R}, 0)/\partial F^n$ and generalized radial functions $P_n(N, F, k)$.

As shown earlier, in the case of uniform initial distribution of concentrations in the external medium, equation (71), not only

$$\theta_e(\bar{R}, 0) = 0, \tag{95}$$

but also all of the higher time derivatives of concentration $T_e(\bar{R}, F)$ vanish at time zero

$$\left[\frac{\partial^n}{\partial F^n} \theta_e(\bar{R}, F) \right]_{F=0} = 0. \tag{96}$$

For the first derivative $\partial \theta_e(\bar{R}, 0)/\partial F$, this follows directly from equation (89), since at $F = 0$ the integral on the RHS is also equal to zero, and subject to equation (95)

$$\nabla^2 \theta_e(\bar{R}, 0) \equiv 0, \tag{97}$$

at the initial instant of time.

For the remaining higher derivatives, this follows from equation (89), both sides of which should be sufficiently differentiated with respect to time, and the initial values of the lower-order derivatives of dimensionless concentration $\theta_e(\bar{R}, F)$ with respect to the Fourier number, F, should be taken into account.

Therefore, the field of dimensionless concentrations in dispersed phase particles is accurately described by the effect component

$$\theta_i(N, F, \bar{R}) = \sum_{n=0}^{\infty} \frac{\partial^n}{\partial F^n} \theta_a(\bar{R}, F) \cdot P_n(N, k), \tag{98}$$

which, under ordinary conditions, corresponds to large values of the Fourier homochronicity complex.

The dimensionless concentration gradient on the particle surface

$$\left[\frac{\partial}{\partial N}\theta_{\mathrm{i}}(N,F,\bar{R})\right]_{N=1} = \sum_{n=1}^{\infty}\frac{\partial^n}{\partial F^n}\theta_{\mathrm{e}}(\bar{R},F)P'_n(1,k) \quad (99)$$

allows one to write the conservation law for the main phase, equation (89)

$$\frac{\partial\theta_{\mathrm{e}}}{\partial F}[1-k\Pi\sigma P'_1(1,k)] - k\Pi\sigma\sum_{n=2}^{\infty}\frac{\partial^n\theta_{\mathrm{e}}}{\partial F^n}P'_n(1,k)\Delta = \nabla^2_{\bar{R}}\theta_{\mathrm{e}}, \quad (100)$$

where [2]

$$P_n(N,k) = \int_1^N \eta^{1-k}\,\mathrm{d}\eta\int_0^\eta P_{n-1}(\xi,k)\xi^{k-1}\,\mathrm{d}\xi;$$

$$P_0(N,k) \equiv 1,$$

as an equation of infinite order with respect to time and of second order with respect to coordinate. For the half-space bounded by a plane, this equation is of the form

$$\frac{\partial\theta_{\mathrm{e}}}{\partial F} - \frac{k\Pi\sigma}{1-\Pi\sigma}\sum_{n=2}^{\infty}P'_n(1,k)\frac{\partial^n\theta_{\mathrm{e}}}{\partial F^n} = \frac{\Delta}{1-\Pi\sigma}\frac{\partial^2\theta_{\mathrm{e}}}{\partial x^2}, \quad (101)$$

where the dimensionless abscissa $x = x/\rho_0$ is proportional to the Cartesian one and where it is taken into account that $P'_1(1,k) = 1/k$. If harmonic oscillations of concentration occur at the boundary of such a half-space

$$\theta_{\mathrm{e}}(0,F) = A\cos\Omega\cdot F, \quad (102)$$

with the dimensionless frequency

$$\Omega = \omega\rho_0^2/D_{\mathrm{i}}, \quad (103)$$

then, having introduced the homochronicity and concentration simplexes

$$H = \omega\cdot\tau = \Omega\cdot F; \quad (104)$$

$$\theta = \theta_{\mathrm{e}}/A, \quad (105)$$

and the complex abscissa

$$\xi = x/\sqrt{(\Omega(1-\Pi\sigma)/\Delta)}, \quad (106)$$

one can present equation (101) in the form

$$\frac{\partial\theta}{\partial H} - \frac{k\Pi\sigma}{1-\Pi\sigma}\sum_{n=2}^{\infty}P'_n(1,k)\Omega^{n-1}\frac{\partial^n\theta}{\partial H^n} = \frac{\partial^2\theta}{\partial\xi^2}. \quad (107)$$

Clearly, for this equation to be reduced to the polyrelaxational equation (11), it is necessary to determine the powers of dimensionless relaxation times as the products

$$R_{n-1}^{n-1} = -\frac{k\Pi\sigma}{1-\Pi\sigma}\Omega^{n-1}P'_n(1,k), \quad (108)$$

after which equation (107) is written as

$$\sum_{n=0}^{\infty}R_n^n\frac{\partial^{n+1}\theta}{\partial H^{n+1}} = \frac{\partial^2\theta}{\partial\xi^2}. \quad (11)$$

Subject to equations (12), which are accurately observed in this case, this equation has an asymptotic solution, equation (13), in which the functions of frequencies and of relaxation time spectra, equations (14) and (15), are realized as the sums of the series

$$\alpha(\Omega) = \sum_{m=1}^{\infty}(-1)^{m-1}R_{2m-1}^{2m-1} = -\frac{k\Pi\sigma}{1-\Pi\sigma} \times [\Omega P'_2(1,k)-\Omega^3P'_4(1,k)+\Omega^5P'_6(1,k)-\cdots], \quad (109)$$

$$\beta(\Omega) = \sum_{m=0}^{\infty}(-1)^m R_{2m}^{2m} = 1-\frac{k\Pi\sigma}{1-\Pi\sigma} \times [\Omega^2P'_3(1,k)-\Omega^4P'_5(1,k)+\Omega^6P'_7(1,k)-\cdots], \quad (110)$$

which are positive and exist at all, arbitrarily large, values of the angular frequency Ω, as is shown in ref. [2].

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APPENDIX

Evaluation of the sum of infinite series (109) and (110)

$$\alpha(\Omega) = -\frac{k\Pi\sigma}{1-\Pi\sigma}\sum_{n=1}^{\infty}(-1)^n\Omega^{2n-1}P'_{2n}(1,k); \tag{A1}$$

$$\beta(\Omega) = 1-\frac{k\Pi\sigma}{1-\Pi\sigma}\sum_{n=1}^{\infty}(-1)^{n-1}\Omega^{2n}P'_{2n+1}(1,k), \tag{A2}$$

is based on the basic properties of the radial polynomials of a symmetric field, as these are presented in ref. [2].

By definition, these polynomials satisfy the following Poisson-type system of equations

$$\nabla^2 P_n \equiv \frac{\mathrm{d}^2 P_n}{\mathrm{d}N^2}+\frac{k-1}{N}\frac{\mathrm{d}P_n}{\mathrm{d}N} = P_{n-1}, \tag{A3}$$

$$n = 1,2,\ldots, N\in[0;\,1]$$

and homogeneous boundary-value conditions of the first and second kind

$$P'_n(0) = 0,\quad P_n(1) = 0, \tag{A4}$$

so that these can be found by successive integration

$$P_n(N,k) = \int_1^N \eta^{1-k}\,\mathrm{d}\eta\int_0^{\eta}\xi^{k-1}P_{n-1}(\xi)\,\mathrm{d}\xi. \tag{A5}$$

Since on the interval $[0;\,1]$ the lowest order polynomial is

$$P_0(N)\equiv 1, \tag{A6}$$

all the polynomials in this interval are sign-reversible

$$\operatorname{sign} P_n(N,k) = (-1)^n,\quad N\in[0;1]. \tag{A7}$$

Since the polynomial derivative at the right end of the interval is

$$P'_n(1,k) = \int_0^1 \xi^{k-1}P_{n-1}(\xi,k)\,\mathrm{d}\xi, \tag{A8}$$

then, according to equation (A7)

$$\operatorname{sign} P'_n(1,k) = -\operatorname{sign} P_{n-1}(N,k) = (-1)^{n-1}. \tag{A9}$$

Specifically, depending on the k number of finite measurements, the following relations take place

$$\begin{aligned}
&P'_1(1,k) = k^{-1};\\
&P'_2(1,k) = -k^{-2}(k+2)^{-1};\\
&P'_3(1,k) = (k^2+8k+32)\cdot 8^{-1}k^{-3}(k+2)^{-1}(k+4)^{-1};\\
&P'_4(1,k) = -(5k+2)k^{-4}(k+2)^{-2}(k+4)^{-1}(k+6)^{-1}.
\end{aligned} \tag{A10}$$

In order to evaluate the sums $\alpha(\Omega)$ and $\beta(\Omega)$, consider the sign-reversible series

$$u(N,\Omega,k) = 1+\Omega P_1(N,k)+\Omega^2P_2(N,k)+\Omega^3P_3(N,k)+\cdots, \tag{A11}$$

which converges at $N\in[0;1]$ and at

$$|\Omega|<2k, \tag{A12}$$

since the radial polynomials are limited by the terms of geometric progression

$$|P_n(N,k)|<(2k)^{-n},\quad N\in[0;1]. \tag{A13}$$

In view of system (A3), the function $U(N,\Omega,k)$ satisfies the equation

$$\nabla^2U(N;\Omega,k)-\Omega U(N;\Omega,k) = 0, \tag{A14}$$

and, in consequence of equations (A4), the boundary conditions

$$U'(0;\Omega,k) = 0,\quad U(1;\Omega,k) = 1. \tag{A15}$$

Depending on k, it is

$$U(N;\Omega,1) = \cosh N\sqrt{\Omega}/\cosh\sqrt{\Omega}, \tag{A16}$$

$$U(N;\Omega,2) = I_0(N\sqrt{\Omega})/I_0(\sqrt{\Omega}); \tag{A17}$$

$$U(N;\Omega,3) = \sinh N\sqrt{\Omega}/[N\sinh\sqrt{\Omega}], \tag{A18}$$

and can be calculated on the interval $N\in[0;1]$ at arbitrarily large frequencies Ω. Therefore, the function $U(N;\Omega,k)$ represents the sum of the series (A11) at all $\Omega\in[0,\infty]$ including also $\Omega\gg 2k$.

Since

$$U(N;-\Omega,k) = 1-\Omega P_1+\Omega^2P_2-\Omega^3P_3+\cdots, \tag{A19}$$

then the even component of the function (A11) is

$$U_c(N;\Omega,k) = \tfrac{1}{2}[U(N;\Omega,k)+U(N;-\Omega,k)], \tag{A20}$$

while the odd component is

$$U_s(N;\Omega,k) = \tfrac{1}{2}[U(N;\Omega,k)-U(N;-\Omega,k)]. \tag{A21}$$

If now the real number Ω is replaced by the imaginary one $\mathrm{i}\Omega$, then both components will pass into the sign-reversible series

$$U_c(N;\mathrm{i}\,\Omega,k) = 1-\Omega^2P_2+\Omega^4P_4-\Omega^6P_6+\cdots, \tag{A22}$$

$$U_s(N;\mathrm{i}\,\Omega,k) = \mathrm{i}\,[\Omega P_1-\Omega^3P_3+\Omega^5P_5-\Omega^7P_7+\cdots]. \tag{A23}$$

It is relatively simple to change from these functions to the functions $\alpha(\Omega)$, equation (109), and $\beta(\Omega)$, equation (110), by performing differentiation with respect to the radius at the point $N = 1$. Since equality (A8) is valid, then

$$U'_c(1;\mathrm{i}\,\Omega k) = -\Omega[\Omega P'_2(1,k)-\Omega^3P'_4(1,k)+\Omega^5P'_6(1,k)-\cdots] \tag{A24}$$

and

$$\alpha(\Omega) = \frac{k\Pi\sigma}{1-\Pi\sigma}\frac{1}{\Omega}U'_c(1;\mathrm{i}\,\Omega,k). \tag{A25}$$

The sum of the even powers

$$\Omega^2P_3(N,k)-\Omega^4P_5(N,k)+\Omega^6P_7(N,k)-\cdots = \frac{\Omega P_1(N,k)+\mathrm{i}\,U_s(N,\Omega,k)}{\Omega}, \tag{A26}$$

and differentiation yields the relation

$$\Omega^2P'_3(1,k)-\Omega^4P'_5(1,k)+\Omega^6P'_7(1,k)-\cdots = \frac{1}{k}+\frac{\mathrm{i}}{\Omega}U'_s(1;\mathrm{i}\,\Omega,k). \tag{A27}$$

Consequently

$$\beta(\Omega) = 1-\frac{k\Pi\sigma}{1-\Pi\sigma}\left[\frac{1}{k}+\frac{\mathrm{i}}{\Omega}U'_s(1;\mathrm{i}\,\Omega,k)\right]. \tag{A28}$$

For spherical particles, $k = 3$ and

$$U(N;\Omega,3) = \frac{\sinh N\sqrt{\Omega}}{N\sinh\sqrt{\Omega}}. \tag{A29}$$

Obviously, the even and odd components have the following form

$$\begin{aligned}
U_c(N,\Omega,3) &= \frac{1}{2}\left[\frac{\sinh N\sqrt{\Omega}}{N}+\frac{\sinh N\sqrt{-\Omega}}{N\sinh\sqrt{-\Omega}}\right]\\
&= \frac{1}{2}\left[\frac{\sinh N\sqrt{\Omega}}{N\sinh\sqrt{\Omega}}-\frac{\sin N\sqrt{\Omega}}{N\sin\sqrt{\Omega}}\right],
\end{aligned} \tag{A30}$$

$$U_s(N;\Omega,3) = \frac{1}{2}\left[\frac{\sinh N\sqrt{\Omega}}{N\sinh\sqrt{\Omega}}-\frac{\sin N\sqrt{\Omega}}{N\sin\sqrt{\Omega}}\right]. \tag{A31}$$

The derivatives of these functions

$$U_c'(N;\Omega,3)=\frac{1}{2N^2}\left[\frac{N\sqrt{\Omega}\cosh N\sqrt{\Omega}-\sinh N\sqrt{\Omega}}{\sinh\sqrt{\Omega}}+\frac{N\sqrt{\Omega}\cos N\sqrt{\Omega}-\sin N\sqrt{\Omega}}{\sin\sqrt{\Omega}}\right], \quad \text{(A32)}$$

$$U_s'(N,\Omega,3)=\frac{1}{2N^2}\left[\frac{N\sqrt{\Omega}\cosh N\sqrt{\Omega}-\sinh\sqrt{\Omega}}{\sinh\sqrt{\Omega}}\times\frac{N\sqrt{\Omega}\cos N\sqrt{\Omega}-\sin N\sqrt{\Omega}}{\sin\sqrt{\Omega}}\right], \quad \text{(A33)}$$

at the point $N = 1$ take on the values

$$U_c'(1;\Omega,3)=\tfrac{1}{2}[\sqrt{\Omega}(\coth\sqrt{\Omega}+\cot\sqrt{\Omega})-2], \quad \text{(A34)}$$

$$U_s'(1;\Omega,3)=\tfrac{1}{2}[\sqrt{\Omega}(\coth\sqrt{\Omega}-\cot\sqrt{\Omega})]. \quad \text{(A35)}$$

In the case of a purely imaginary value of the argument, the following relations are valid

$$\sqrt{(\mathrm{i}\,\Omega)}\,[\coth\sqrt{(\mathrm{i}\,\Omega)}+\cot\sqrt{(\mathrm{i}\,\Omega)}]=\sqrt{\left(\frac{\Omega}{2}\right)}\frac{\sin\sqrt{(2\Omega)}+\sinh\sqrt{(2\Omega)}}{\sin^2\sqrt{(\Omega/2)}+\sinh^2\sqrt{(\Omega/2)}}, \quad \text{(A36)}$$

$$\sqrt{(\mathrm{i}\,\Omega)}\,[\coth\sqrt{(\mathrm{i}\,\Omega)}-\cot\sqrt{(\mathrm{i}\,\Omega)}]=\mathrm{i}\sqrt{\left(\frac{\Omega}{2}\right)}\frac{\sinh\sqrt{(2\Omega)}-\sin\sqrt{(2\Omega)}}{\sinh^2\sqrt{(\Omega/2)}+\sin^2\sqrt{(\Omega/2)}} \quad \text{(A37)}$$

and functions (A34) and (A35) take on the values

$$U_c'(1;\mathrm{i}\,\Omega,3)=\frac{1}{2}\sqrt{\left(\frac{\Omega}{2}\right)}\frac{\sinh\sqrt{(2\Omega)}-\sin\sqrt{(2\Omega)}}{\sinh^2\sqrt{(\Omega/2)}+\sin^2\sqrt{(\Omega/2)}}-1, \quad \text{(A38)}$$

$$U_s'(1;\mathrm{i}\,\Omega,3)=\frac{\mathrm{i}}{2}\sqrt{\left(\frac{\Omega}{2}\right)}\frac{\sinh\sqrt{(2\Omega)}-\sin\sqrt{(2\Omega)}}{\sinh^2\sqrt{(\Omega/2)}+\sin^2\sqrt{(\Omega/2)}}. \quad \text{(A39)}$$

It is clear that the functions of the frequency for a spherical particle

$$\alpha(\Omega)=\frac{3\Pi\sigma}{1-\Pi\sigma}\left[\frac{1}{2\sqrt{(2\Omega)}}\frac{\sinh\sqrt{(2\Omega)}+\sin\sqrt{(2\Omega)}}{\sinh^2\sqrt{(\Omega/2)}+\sin^2\sqrt{(\Omega/2)}}-\frac{1}{\Omega}\right], \quad \text{(A40)}$$

$$\beta(\Omega)=\frac{1-2\Pi\sigma}{1-\Pi\sigma}+\frac{1.5\Pi\sigma}{1-\Pi\sigma}\frac{1}{\sqrt{(2\Omega)}}\frac{\sinh\sqrt{(2\Omega)}-\sin\sqrt{(2\Omega)}}{\sinh^2\sqrt{(\Omega/2)}+\sin^2\sqrt{(\Omega/2)}}, \quad \text{(A41)}$$

at large frequencies have the final limits 0 and $(1-2\Pi\sigma)/(1-\Pi\sigma)$.

The function

$$\gamma(\Omega)=\alpha(\Omega)+\sqrt{(\alpha^2(\Omega)+\beta^2(\Omega))}, \quad \text{(A42)}$$

at large Ω's has the limit

$$\lim_{\Omega\to\infty}\gamma(\Omega)=\frac{1-2\Pi\sigma}{1-\Pi\sigma}, \quad \text{(A43)}$$

and, therefore, the phase velocity of propagation of vibrations in such a nonhomogeneous medium increases infinitely with frequency.

HETERODIFFUSION ET POLYRELAXATION

Résumé—Dans un mélange du milieu principal avec un milieu dispersé, les coefficients de diffusion de masse ou d'énergie diffèrent sensiblement, les échanges par une substance diffusante se produisent durant le mécanisme global. Aussi n'est-il pas possible de prendre en moyenne les coefficients de diffusion pour un mélange hétérogène. Le transport de la substance diffusante est décrite par une équation de type polyrelaxation avec un spectre infini de temps de relaxation. Trois types de milieux polyrelaxationnels sont trouvés dans lesquels la vitesse de phase des vibrations à haute fréquence croît infiniment, a une valeur constante ou approche zéro avec un accroissement de fréquence angulaire, et par exemple il se produit une trempe de telles oscillations.

HETERODIFFUSION UND POLYRELAXATION

Zusammenfassung—Bei einer Mischung aus einer Grundsubstanz und einem dispersen Füllstoff, dessen Diffusionskoeffizienten für Massen- und Energietransport sich wesentlich unterscheiden, findet ein Austausch durch eine diffundierende Substanz während des gesamten Prozesses statt. Die Mittelung der Diffusionskoeffizienten für ein heterogenes Gemisch wie dieses ist unmöglich. Der Stofftransport infolge Diffusion wird durch eine Gleichung vom Polyrelaxationstyp mit einem unendlichen Spektrum von Relaxationszeiten beschrieben. Es wurden 3 Arten von Mehrfachrelaxationsmedien gefunden, bei denen die Phasengeschwindigkeit der Fortpflanzung von hochfrequenten Vibrationen über alle Grenzen ansteigt, einen konstanten Wert erreicht oder mit der Zunahme der Winkelfrequenz gegen null geht, d. h. daß solche Oszillationen unterdrückt werden.

ГЕТЕРОДИФФУЗИЯ И ПОЛИРЕЛАКСАЦИЯ

Аннотация—В смеси основной среды и дисперсного наполнителя, для которых коэффициенты диффузии массы или энергии существенно различны, обмен диффундирующей субстанцией имеет место в течение всего процесса. Для такой гетерогенной смеси невозможно осреднение коэффициента диффузии. Перенос диффундирующей субстанции описывается уравнением полирелаксационного типа с бесконечным спектром времен релаксации. Установлены три типа полирелаксационных сред, в которых фазовая скорость распространения высокочастотных колебаний бесконечно увеличивается, имеет постоянное значение или приближается к нулю с ростом угловой частоты, т.е. такие осцилляции гасятся.